CBSE 2015 FOREIGN EXAMINATION

(Series SSO Code No. 65/1/F, 65/2/F, 65/3/F : Foreign Region)

Note that all the sets have same questions. Only their sequence of appearance is different.

Max. Marks : 100 **Time Allowed : 3 Hours**

SECTION – A

Q01. Find the area of a parallelogram whose adjacent sides are represented by the vectors $2\hat{i}-3\hat{k}$ and $4\hat{j}+2\hat{k}$.

Sol. Let $\vec{a}=2\hat{i}-3\hat{k}$ and $\vec{b}=4\hat{j}+2\hat{k}$. Now, $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 0 & -3\\ 0 & 4 & 2\end{vmatrix}=12\hat{i}-4\hat{j}+8\hat{k}$

$\therefore$ required area $=\left|\vec{a}\times\vec{b}\right|=\left|12\hat{i}-4\hat{j}+8\hat{k}\right|=\sqrt{144+16+64}=\sqrt{224}=4\sqrt{14}$ sq.units

Q02. Find the sum of the intercepts cut off by the plane $2x+y-z=5$ on the coordinate axes.

Sol. Given plane $2x+y-z=5$ i.e., $\frac{x}{5/2}+\frac{y}{5}+\frac{z}{-5}=1$

On comparing to $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, we have x, y and z-intercepts as 5/2, 5, –5 respectively.

$\therefore$ sum of the intercepts $=\frac{5}{2}+5+(-5)=\frac{5}{2}$.

Q03. Find the unit vector in the direction of the sum of the vectors $2\hat{i}+3\hat{j}-\hat{k}$ and $4\hat{i}-3\hat{j}+2\hat{k}$.

Sol. Let $\vec{p}=2\hat{i}+3\hat{j}-\hat{k}+4\hat{i}-3\hat{j}+2\hat{k}=6\hat{i}+\hat{k}$

$\therefore$ required unit vector, $\hat{p}=\frac{\vec{p}}{|\vec{p}|}=\frac{6\hat{i}+\hat{k}}{\sqrt{36+1}}=\frac{1}{\sqrt{37}}[6\hat{i}+\hat{k}]$.

Q04. Write the sum of the order and degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{dy}{dx}\right)^3+x^4=0$.

Sol. Order and degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{dy}{dx}\right)^3+x^4=0$ is 2 and 2 respectively.

$\therefore$ required sum of order and degree $=2+2=4$.

Q05. Write the solution of the differential equation $\frac{dy}{dx}=2^{-y}$.

Sol. Given $\frac{dy}{dx}=2^{-y}$ $\Rightarrow \frac{dy}{2^{-y}}=dx$

$\Rightarrow \int 2^y dy=\int dx$ $\therefore \frac{2^y}{\log_e 2}=x+C$ or, $2^y\log_2 e=x+C$

Q06. If $A=\begin{bmatrix}5 & 6 & -3\\ -4 & 3 & 2\\ -4 & -7 & 3\end{bmatrix}$, then write the cofactor of the element a_{21} of its 2nd row.

Sol. Cofactor of element a_{21}, $C_{21}=-[6\times3-(-7)(-3)]=3$.

SECTION – B

Q07. Find the point on the curve $9y^2=x^3$, where the normal to the curve makes equal intercepts on the axes.

Sol. We have $9y^2=x^3$. Let the required point be $P(\alpha,\beta)$. So, $9\beta^2=\alpha^3$...(i)

On differentiating the eq. of curve w.r.t. x, $18y\frac{dy}{dx}=3x^2 \quad \Rightarrow \frac{dy}{dx}=\frac{x^2}{6y}$

$\therefore \left.\frac{dy}{dx}\right]_{\text{at P}}=\frac{\alpha^2}{6\beta}=m_T \qquad \Rightarrow m_N=-\frac{6\beta}{\alpha^2}$

Since the normal makes equal intercepts with the axes, $\therefore m_N=\pm1 \qquad \Rightarrow -\frac{6\beta}{\alpha^2}=\pm1$

That is, $-6\beta=\alpha^2,\ 6\beta=\alpha^2 \qquad \Rightarrow \beta=-\frac{\alpha^2}{6},\ \beta=\frac{\alpha^2}{6}\ldots$(ii)

Solving (i) and (ii) simultaneously, we get : $9\left(-\frac{\alpha^2}{6}\right)^2=\alpha^3,\ 9\left(\frac{\alpha^2}{6}\right)^2=\alpha^3$

$\Rightarrow \alpha^4-4\alpha^3=0,\ \alpha^4-4\alpha^3=0 \qquad \Rightarrow \alpha^3(\alpha-4)=0,\ \alpha^3(\alpha-4)=0 \qquad \therefore \alpha=0,4$

So, $\beta=0,\pm\frac{8}{3}$

Hence the required points are $\left(4,\pm\frac{8}{3}\right)$ and $(0,0)$.

Q08. If $y=[x+\sqrt{1+x^2}]^n$, then show that $(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=n^2y$.

Sol. We've $y=\left[x+\sqrt{1+x^2}\right]^n \qquad \Rightarrow \frac{dy}{dx}=n\left[x+\sqrt{x^2+1}\right]^{n-1}\times\left(1+\frac{2x}{2\sqrt{x^2+1}}\right)$

$\Rightarrow \frac{dy}{dx}=n\frac{\left[x+\sqrt{x^2+1}\right]^n}{\left[x+\sqrt{x^2+1}\right]}\times\left(\frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}\right) \qquad \Rightarrow \frac{dy}{dx}=\frac{ny}{\sqrt{x^2+1}} \qquad \Rightarrow \sqrt{x^2+1}\frac{dy}{dx}=ny$

$\Rightarrow \sqrt{x^2+1}\frac{d^2y}{dx^2}+\frac{dy}{dx}\times\frac{2x}{2\sqrt{x^2+1}}=n\times\frac{dy}{dx} \qquad \Rightarrow (x^2+1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=n\sqrt{x^2+1}\frac{dy}{dx}$

$\Rightarrow (x^2+1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=n\times ny \qquad \therefore (1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=n^2y$.

Q09. Find whether the following function is differentiable at $x=1$ and $x=2$ or not :

$$f(x)=\begin{cases} x, & x<1 \\ 2-x, & 1\le x\le 2 \\ -2+3x-x^2, & x>2 \end{cases}.$$

Sol. We have $Lf'(1)=\lim_{x\to1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to1^-}\frac{x-(2-1)}{x-1}=\lim_{x\to1^-}\frac{x-1}{x-1}=1$,

$Rf'(1)=\lim_{x\to1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to1^+}\frac{2-x-(2-1)}{x-1}=\lim_{x\to1^+}\frac{-(x-1)}{x-1}=-1\ne Lf'(1)$

Hence $f(x)$ isn't differentiable at $x=1$.

$Lf'(2)=\lim_{x\to2^-}\frac{f(x)-f(2)}{x-2}=\lim_{x\to2^-}\frac{2-x-(2-2)}{x-2}=\lim_{x\to2^-}\frac{-(x-2)}{x-2}=-1$,

$Rf'(2)=\lim_{x\to2^+}\frac{f(x)-f(2)}{x-2}=\lim_{x\to2^+}\frac{-2+3x-x^2-(2-2)}{x-2}=\lim_{x\to2^+}\frac{-(x-2)(x-1)}{x-2}$

$\Rightarrow \quad =\lim_{x\to2^+}-(x-1)=-(2-1)=-1=Lf'(2)$

Hence $f(x)$ is differentiable at $x=2$.

Q10. In a parliament election, a political party hired a public relation firm to promote its candidates in 3 ways – telephone, house calls and letters. The cost per contact (in paise) is given in matrix A as

$$A = \begin{bmatrix} 140 \\ 200 \\ 150 \end{bmatrix} \begin{matrix} \text{Telephone} \\ \text{House Call} \\ \text{Letters} \end{matrix}$$

The number of contacts of each type made in two cities X and Y is given in the matrix B as

$$\begin{matrix} \quad\text{Telephone} & \text{House Call} & \text{Letters} \end{matrix}$$
$$B = \begin{bmatrix} 1000 & 500 & 5000 \\ 3000 & 1000 & 10000 \end{bmatrix} \begin{matrix} \text{City X} \\ \text{City Y} \end{matrix}$$

Find the total amount spent by the party in the two cities.

What should one consider before casting his/her vote – party's promotional activity or their social activities?

Sol. The cost per contact (in paise) is given in matrix A as $A = \begin{bmatrix} 140 \\ 200 \\ 150 \end{bmatrix} \begin{matrix} \text{Telephone} \\ \text{House Call} \\ \text{Letters} \end{matrix}$

The number of contacts of each type made in two cities X and Y is given in the matrix B as

$$\begin{matrix} \quad\text{Telephone} & \text{House Call} & \text{Letters} \end{matrix}$$
$$B = \begin{bmatrix} 1000 & 500 & 5000 \\ 3000 & 1000 & 10000 \end{bmatrix} \begin{matrix} \text{City X} \\ \text{City Y} \end{matrix}$$

The total amount spent by the party in the two cities $= BA$

That is, $$= \begin{bmatrix} 1000 & 500 & 5000 \\ 3000 & 1000 & 10000 \end{bmatrix} \begin{bmatrix} 140 \\ 200 \\ 150 \end{bmatrix} = 500 \begin{bmatrix} 2 & 1 & 10 \\ 6 & 2 & 20 \end{bmatrix} \left(10 \begin{bmatrix} 14 \\ 20 \\ 15 \end{bmatrix} \right)$$

$$\Rightarrow \quad = 5000 \begin{bmatrix} 2 & 1 & 10 \\ 6 & 2 & 20 \end{bmatrix} \begin{bmatrix} 14 \\ 20 \\ 15 \end{bmatrix} = 5000 \begin{bmatrix} 28+20+150 \\ 84+40+300 \end{bmatrix} = 5000 \begin{bmatrix} 198 \\ 424 \end{bmatrix}$$

$$\Rightarrow \quad = \begin{bmatrix} 990000 \\ 2120000 \end{bmatrix} \begin{matrix} \leftarrow \text{Amount spent in City X (in paise)} \\ \leftarrow \text{Amount spent in City Y (in paise)} \end{matrix}$$

Hence, the party spent 990000 paise (or, ₹9900) in the City X and, 2120000 paise (or, ₹21200) in the City Y.

One should consider party's **social activities** instead of promotional activities of the party before casting his/her vote.

Q11. Evaluate : $\int e^{2x}.\sin(3x+1)dx$.

Sol. Let $I = \int e^{2x}.\sin(3x+1)dx \ldots(i)$

Applying integral By Parts, we get :

$$I = \sin(3x+1)\int e^{2x}dx - \int\left(\frac{d}{dx}\sin(3x+1)\int e^{2x}dx\right)dx$$

$$\Rightarrow I = \sin(3x+1) \times \frac{e^{2x}}{2} - \int 3\cos(3x+1) \times \frac{e^{2x}}{2}dx$$

$$\Rightarrow I = \frac{e^{2x}}{2}\sin(3x+1) - \frac{3}{2}\int e^{2x}.\cos(3x+1)dx$$

$$\Rightarrow I=\frac{e^{2x}}{2}\sin(3x+1)-\frac{3}{2}\left[\cos(3x+1)\int e^{2x}dx-\int\left(\frac{d}{dx}\cos(3x+1)\int e^{2x}dx\right)dx\right]$$

$$\Rightarrow I=\frac{e^{2x}}{2}\sin(3x+1)-\frac{3}{2}\left[\cos(3x+1)\times\frac{e^{2x}}{2}-\int -3\sin(3x+1)\times\frac{e^{2x}}{2}dx\right]$$

$$\Rightarrow I=\frac{e^{2x}}{2}\sin(3x+1)-\frac{3}{2}\left[\frac{e^{2x}}{2}\cos(3x+1)+\frac{3}{2}\int e^{2x}.\sin(3x+1)dx\right]$$

$$\Rightarrow I=\frac{e^{2x}}{2}\sin(3x+1)-\frac{3e^{2x}}{4}\cos(3x+1)-\frac{9}{4}I \qquad \text{[By (i)}$$

$$\Rightarrow I+\frac{9}{4}I=\frac{e^{2x}}{4}[2\sin(3x+1)-3\cos(3x+1)] \qquad \therefore I=\frac{e^{2x}}{13}[2\sin(3x+1)-3\cos(3x+1)]+C$$

Q12. Evaluate : $\int_{-\pi/2}^{\pi/2}\frac{\cos x}{1+e^x}dx$.

Sol. Consider $I=\int_{-\pi/2}^{\pi/2}\frac{\cos x}{1+e^x}dx$...(i)

Using $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$, we get

$$I=\int_{-\pi/2}^{\pi/2}\frac{\cos\left(-\frac{\pi}{2}+\frac{\pi}{2}-x\right)}{1+e^{-\pi/2+\pi/2-x}}dx=\int_{-\pi/2}^{\pi/2}\frac{\cos x}{1+e^{-x}}dx \qquad \Rightarrow I=\int_{-\pi/2}^{\pi/2}\frac{e^x\cos x}{e^x+1}dx\text{...(ii)}$$

On adding (i) & (ii), we get : $2I=\int_{-\pi/2}^{\pi/2}\frac{\cos x}{1+e^x}dx+\int_{-\pi/2}^{\pi/2}\frac{e^x\cos x}{e^x+1}dx$

$$\Rightarrow 2I=\int_{-\pi/2}^{\pi/2}\left(\frac{1}{1+e^x}+\frac{e^x}{e^x+1}\right)\cos x dx \qquad \Rightarrow 2I=\int_{-\pi/2}^{\pi/2}\cos x dx$$

$$\Rightarrow 2I=2\int_0^{\pi/2}\cos x dx \qquad \left[\begin{array}{l}\because f(x)=\cos x=\cos(-x)=f(-x),\text{ i.e. } f \text{ is even function}\\ \text{and, } \int_{-a}^{a}f(x)dx=2\int_0^a f(x)dx, \text{ if } f \text{ is an even function}\end{array}\right.$$

$$\therefore I=[\sin x]_0^{\pi/2}=\sin\frac{\pi}{2}-\sin 0=1.$$

Q13. Three machines E_1, E_2 and E_3 in a certain factory producing electric bulbs, produces 50%, 25% and 25% respectively, of the total daily output of electric bulbs. It is known that 4% of the bulbs produced by each of machines E_1 and E_2 are defective and that 5% of those produced by machine E_3 are defective. If one bulb is picked up at random from a day's production, calculate the probability that it is defective.

OR Two numbers are selected at random (without replacement) from positive integers 2, 3, 4, 5, 6, and 7. Let X denote the larger of the two numbers obtained. Find the mean and variance of the probability distribution of X.

Sol. Let E_1, E_2 and E_3 denote the events that bolts produced by machines E_1, E_2 and E_3 respectively. Let A be the event that the selected bulb is defective.

$$\therefore P(E_1)=\frac{50}{100}=\frac{1}{2},\ P(E_2)=P(E_3)=\frac{25}{100}=\frac{1}{4},\ P(A|E_1)=P(A|E_2)=\frac{4}{100}=\frac{1}{25},$$

and $P(A|E_3)=\frac{5}{100}=\frac{1}{20}$

So, $P(A)=P(E_1)P(A|E_1)+P(E_2)P(A|E_2)+P(E_3)P(A|E_3)=\frac{1}{2}\times\frac{1}{25}+\frac{1}{4}\times\frac{1}{25}+\frac{1}{4}\times\frac{1}{20}$

Therefore, $P(A)=\frac{17}{400}$ or, $\left(\frac{17}{4}\right)\%$.

OR Since X denotes the larger of the two numbers obtained from 2, 3, 4, 5, 6 and 7.
So values of X : 3, 4, 5, 6, 7.

X	P(X)
3	$\frac{1}{6}\times\frac{1}{5}+\frac{1}{6}\times\frac{1}{5}=\frac{1}{15}$
4	$2\left(\frac{1}{6}\times\frac{1}{5}+\frac{1}{6}\times\frac{1}{5}\right)=\frac{2}{15}$
5	$2\left(\frac{1}{6}\times\frac{1}{5}+\frac{1}{6}\times\frac{1}{5}+\frac{1}{6}\times\frac{1}{5}\right)=\frac{3}{15}$
6	$2\left(\frac{1}{6}\times\frac{1}{5}+\frac{1}{6}\times\frac{1}{5}+\frac{1}{6}\times\frac{1}{5}+\frac{1}{6}\times\frac{1}{5}\right)=\frac{4}{15}$
7	$2\left(\frac{1}{6}\times\frac{1}{5}+\frac{1}{6}\times\frac{1}{5}+\frac{1}{6}\times\frac{1}{5}+\frac{1}{6}\times\frac{1}{5}+\frac{1}{6}\times\frac{1}{5}\right)=\frac{5}{15}$

Now, mean $=\sum X\,P(X)=3\times\frac{1}{15}+4\times\frac{2}{15}+5\times\frac{3}{15}+6\times\frac{4}{15}+7\times\frac{5}{15}=\frac{85}{15}$

And variance $=\sum X^2\,P(X)-[\text{Mean}]^2=3^2\times\frac{1}{15}+4^2\times\frac{2}{15}+5^2\times\frac{3}{15}+6^2\times\frac{4}{15}+7^2\times\frac{5}{15}=\frac{14}{9}$

Q14. The two vectors $\hat{j}+\hat{k}$ and $3\hat{i}-\hat{j}+4\hat{k}$ represent the two sides vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ respectively of triangle ABC. Find the length of the median through A.

Sol. Here $\overrightarrow{AB}=\hat{j}+\hat{k},\ \overrightarrow{AC}=3\hat{i}-\hat{j}+4\hat{k}$

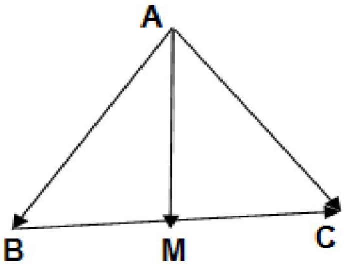
$\because \overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}\quad \therefore \overrightarrow{BC}=(3\hat{i}-\hat{j}+4\hat{k})-(\hat{j}+\hat{k})=3\hat{i}-2\hat{j}+3\hat{k}$

Since median through A meets the side BC at the midpoint of BC.

If M is the mid-point of $\overrightarrow{BC}$ then, $\overrightarrow{BM}=\frac{3}{2}\hat{i}-\hat{j}+\frac{3}{2}\hat{k}$

Therefore $\overrightarrow{AM}=\overrightarrow{AB}+\overrightarrow{BM}=\hat{j}+\hat{k}+\frac{3}{2}\hat{i}-\hat{j}+\frac{3}{2}\hat{k}=\frac{3}{2}\hat{i}+\frac{5}{2}\hat{k}$

$\therefore$ Length of median, $|\overrightarrow{AM}|=\sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{5}{2}\right)^2}=\frac{\sqrt{34}}{2}$ units



Q15. Find the equation of a plane which passes through the point (3, 2, 0) and contains the line $\frac{x-3}{1}=\frac{y-6}{5}=\frac{z-4}{4}$.

Sol. The equation of plane through (3, 2, 0) is $A(x-3)+B(y-2)+C(z-0)=0\ldots(i)$ where A, B, C are the d.r.'s of the normal to the required plane.

As plane (i) contains the line $\frac{x-3}{1}=\frac{y-6}{5}=\frac{z-4}{4}$ with d.r.'s as 1, 5, 4 so,

$A+5B+4C=0\ldots(ii)$.

Also (3, 6, 4) lies on the given line and plane (i) as well so, $A(3-3)+B(6-2)+C(4-0)=0$

i.e., $0.A+4B+4C=0\Rightarrow 0A+B+C=0\ldots(iii)$

Solving (ii) and (iii), we get $\frac{A}{1}=\frac{B}{-1}=\frac{C}{1}$ i.e., d.r.'s of the normal to plane (i) are 1, –1, 1.

Hence equation of plane is : $1.(x-3)-1.(y-2)+1.z=0$ i.e., $x-y+z=1$.

Q16. If $2\tan^{-1}(\cos\theta)=\tan^{-1}(2\operatorname{cosec}\theta),\ (\theta\neq 0)$, then find the value of θ.

OR If $\tan^{-1}\left(\frac{1}{1+1.2}\right)+\tan^{-1}\left(\frac{1}{1+2.3}\right)+...+\tan^{-1}\left(\frac{1}{1+n.(n+1)}\right)=\tan^{-1}\theta$, then find the value of θ.

Sol. Given $2\tan^{-1}(\cos\theta)=\tan^{-1}(2\operatorname{cosec}\theta)$ $\Rightarrow \tan^{-1}\frac{2\cos\theta}{1-\cos^2\theta}=\tan^{-1}(2\operatorname{cosec}\theta)$

$\Rightarrow \tan\tan^{-1}\frac{2\cos\theta}{1-\cos^2\theta}=\tan\tan^{-1}\frac{2}{\sin\theta}$ $\Rightarrow \frac{2\cos\theta}{\sin^2\theta}=\frac{2}{\sin\theta}$ $\Rightarrow \sin\theta\cos\theta-\sin^2\theta=0$

$\Rightarrow \sin\theta[\cos\theta-\sin\theta]=0$ $\Rightarrow \sin\theta=0$ or $\cos\theta-\sin\theta=0$

$\Rightarrow \theta=0$ or $\cot\theta=1$ $\Rightarrow \theta=0$ or $\theta=\frac{\pi}{4}$ But $\theta\neq 0$ $\therefore \theta=\frac{\pi}{4}$.

OR We have $\tan^{-1}\left(\frac{1}{1+1.2}\right)+\tan^{-1}\left(\frac{1}{1+2.3}\right)+...+\tan^{-1}\left(\frac{1}{1+n.(n+1)}\right)=\tan^{-1}\theta$

$\Rightarrow \tan^{-1}\left(\frac{2-1}{1+1.2}\right)+\tan^{-1}\left(\frac{3-2}{1+2.3}\right)+...+\tan^{-1}\left(\frac{(n+1)-n}{1+n.(n+1)}\right)=\tan^{-1}\theta$

$\Rightarrow \tan^{-1}2-\tan^{-1}1+\tan^{-1}3-\tan^{-1}2+...+\tan^{-1}n-\tan^{-1}(n-1)+\tan^{-1}(n+1)-\tan^{-1}n=\tan^{-1}\theta$

$\Rightarrow \tan^{-1}(n+1)-\tan^{-1}1=\tan^{-1}\theta$ $\Rightarrow \tan^{-1}\frac{(n+1)-1}{1+(n+1).1}=\tan^{-1}\theta$

$\Rightarrow \tan\tan^{-1}\frac{n}{1+n+1}=\tan\tan^{-1}\theta$ $\therefore \theta=\frac{n}{n+2}$.

Q17. If $A=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}$ and I is the identity matrix of order 2, then show that $A^2=4A-3I$. Hence find A^{-1}.

OR If $A=\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}$ and $B=\begin{bmatrix}a & 1\\ b & -1\end{bmatrix}$ and $(A+B)^2=A^2+B^2$, then find the values of a and b.

Sol. We have $A=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}$ $\Rightarrow A^2=A.A=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}=\begin{bmatrix}5 & -4\\ -4 & 5\end{bmatrix}$ ...(i)

Also $4A-3I=4\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}-3\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}8 & -4\\ -4 & 8\end{bmatrix}-\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}=\begin{bmatrix}5 & -4\\ -4 & 5\end{bmatrix}$ ...(ii)

By (i) and (ii), we get : $A^2=4A-3I$.

Pre-multiplying both sides by A^{-1} we get : $A^{-1}AA=4A^{-1}A-3A^{-1}I$ $\Rightarrow IA=4I-3A^{-1}$

$\Rightarrow 3A^{-1}=4I-A=4\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$ $\therefore A^{-1}=\frac{1}{3}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$.

OR We've $(A+B)^2=A^2+B^2$ $\Rightarrow (A+B)(A+B)=A^2+B^2$

$\Rightarrow A.A+A.B+B.A+B.B=A.A+B.B$ $\Rightarrow A.B=-B.A$

So, $\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}\begin{bmatrix}a & 1\\ b & -1\end{bmatrix}=-\begin{bmatrix}a & 1\\ b & -1\end{bmatrix}\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}$

$\Rightarrow \begin{bmatrix}a-b & 2\\ 2a-b & 3\end{bmatrix}=\begin{bmatrix}-a-2 & a+1\\ 2-b & b-1\end{bmatrix}$ By equality of matrices, we get :

$a-b=-a-2,\ 2=a+1, 2a-b=2-b, 3=b-1$

On solving these equations, we get : $a=1,\ b=4$.

Q18. Using properties of determinants, prove the following :

$$\begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix} = (1-a^3)^2 .$$

Sol. LHS : Let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix}$ By $R_1 \to R_1 - aR_2, R_2 \to R_2 - aR_3$

$\Rightarrow \Delta = \begin{vmatrix} 1-a^3 & 0 & 0 \\ 0 & 1-a^3 & 0 \\ a & a^2 & 1 \end{vmatrix}$ Taking $1-a^3$ common from R_1 and R_2 both

$\Rightarrow \Delta = (1-a^3)^2 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & a^2 & 1 \end{vmatrix}$ Expanding along R_1

$\Rightarrow \Delta = (1-a^3)^2 \left[1(1-0)-0+0\right] = (1-a^3)^2 = \text{RHS}.$

Q19. Evaluate : $\int \frac{\sin(x-a)}{\sin(x+a)} dx$. **OR** Evaluate : $\int \frac{x^2}{(x^2+4)(x^2+9)} dx$.

Sol. Let $I = \int \frac{\sin(x-a)}{\sin(x+a)} dx$ $\Rightarrow I = \int \frac{\sin\left[(x+a)-2a\right]}{\sin(x+a)} dx$

$\Rightarrow I = \int \frac{\sin(x+a)\cos 2a - \cos(x+a)\sin 2a}{\sin(x+a)} dx$ $\Rightarrow I = \int \left[\cos 2a - \sin 2a \cot(x+a)\right] dx$

$\therefore I = x\cos 2a - \sin 2a \log\left|\sin(x+a)\right| + C.$

OR Let $I = \int \frac{x^2}{(x^2+4)(x^2+9)} dx$ $\Rightarrow I = -\frac{4}{5}\int \frac{1}{x^2+4} dx + \frac{9}{5}\int \frac{1}{x^2+9} dx$

$\Rightarrow I = -\frac{4}{5} \times \frac{1}{2} \tan^{-1}\frac{x}{2} + \frac{9}{5} \times \frac{1}{3} \tan^{-1}\frac{x}{3} + C$ or, $I = \frac{3}{5}\tan^{-1}\frac{x}{3} - \frac{2}{5}\tan^{-1}\frac{x}{2} + C.$

SECTION – C

Q20. Solve the given differential equation : $\left[y - x\cos\left(\frac{y}{x}\right)\right]dy + \left[y\cos\left(\frac{y}{x}\right) - 2x\sin\left(\frac{y}{x}\right)\right]dx = 0.$

OR Solve the given differential equation : $\sqrt{1+x^2+y^2+x^2y^2}\,dx + xydy = 0.$

Sol. Given $\left[y - x\cos\left(\frac{y}{x}\right)\right]dy + \left[y\cos\left(\frac{y}{x}\right) - 2x\sin\left(\frac{y}{x}\right)\right]dx = 0$

$\Rightarrow \frac{dy}{dx} = \frac{2x\sin\left(\frac{y}{x}\right) - y\cos\left(\frac{y}{x}\right)}{y - x\cos\left(\frac{y}{x}\right)}$ $\Rightarrow \frac{dy}{dx} = \frac{2\sin\left(\frac{y}{x}\right) - \frac{y}{x}\cos\left(\frac{y}{x}\right)}{\frac{y}{x} - \cos\left(\frac{y}{x}\right)}$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ $\therefore v + x\frac{dv}{dx} = \frac{2\sin v - v\cos v}{v - \cos v}$

$\Rightarrow x\frac{dv}{dx} = \frac{2\sin v - v\cos v}{v - \cos v} - v$ $\Rightarrow x\frac{dv}{dx} = \frac{2\sin v - v\cos v - v^2 + v\cos v}{v - \cos v}$

$\Rightarrow \int \frac{v - \cos v}{2\sin v - v^2} dv = \int \frac{dx}{x}$ $\Rightarrow -\frac{1}{2}\int \frac{2\cos v - 2v}{2\sin v - v^2} dv = \int \frac{dx}{x}$

$\Rightarrow -\frac{1}{2}\log\left|2\sin v - v^2\right| = \log|x| + \log|C|$ $\Rightarrow \log\left|\frac{1}{\sqrt{2\sin v - v^2}}\right| = \log|xC|$

$\Rightarrow \left|\frac{1}{\sqrt{2\sin\frac{y}{x} - \frac{y^2}{x^2}}}\right| = \log|xC|$ $\Rightarrow \frac{1}{C^2} = x^2\left(2\sin\frac{y}{x} - \frac{y}{x^2}^2\right)$

$\Rightarrow \frac{1}{C^2} = \left(2x^2\sin\frac{y}{x} - y^2\right)$ or, $y^2 = 2x^2\sin\frac{y}{x} + \lambda$ where $\lambda = -\frac{1}{C^2}$.

OR Given $\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0$ $\Rightarrow \sqrt{(1+x^2)(1+y^2)} + xy\frac{dy}{dx} = 0$

$\Rightarrow \int\frac{\sqrt{1+x^2}}{x}dx + \int\frac{y}{\sqrt{1+y^2}}dy = 0$ $\Rightarrow I_1 + \frac{1}{2}\int\frac{2y}{\sqrt{1+y^2}}dy = 0 ...(i)$

Consider $I_1 = \int\frac{\sqrt{1+x^2}}{x}dx$ Put $x^2+1 = t^2 \Rightarrow dx = \frac{tdt}{\sqrt{t^2-1}}$

$\therefore I_1 = \int\frac{t}{\sqrt{t^2-1}}\frac{t}{\sqrt{t^2-1}}dt$ $\Rightarrow I_1 = \int\frac{t^2}{t^2-1}dt$ $\Rightarrow I_1 = \int\left(1+\frac{1}{t^2-1}\right)dt$

$\Rightarrow I_1 = t + \frac{1}{2}\log\left|\frac{t-1}{t+1}\right|$ $\Rightarrow I_1 = \sqrt{x^2+1} + \frac{1}{2}\log\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right|$

Substituting the value of I_1 in (i), we get : $\sqrt{x^2+1} + \frac{1}{2}\log\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right| + \frac{1}{2}\left[2\sqrt{1+y^2}\right] = C$

$\therefore \frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| + \sqrt{1+x^2} + \sqrt{1+y^2} = C$ is the required solution.

Q21. Find the probability distribution of the number of doublets in four throws of a pair of dice. Also find the mean and variance of this distribution.

Sol. Let E : getting a doublet on the pair of dice $\therefore P(E) = \frac{1}{6}$, $P(\bar{E}) = \frac{5}{6}$.

Let X :Number of doublets in four throws of a pair of dice. So values of X are 0, 1, 2, 3, 4.

X	0	1	2	3	4
P(X)	$\left(\frac{5}{6}\right)^4 = \frac{625}{1296}$	${}^4C_1\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^3 = \frac{500}{1296}$	${}^4C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2 = \frac{150}{1296}$	${}^4C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^1 = \frac{20}{1296}$	$\left(\frac{1}{6}\right)^4 = \frac{1}{1296}$

Mean $= \sum X\ P(X) = 0\times\frac{625}{1296} + 1\times\frac{500}{1296} + 2\times\frac{150}{1296} + 3\times\frac{20}{1296} + 4\times\frac{1}{1296} = \frac{864}{1296} = \frac{2}{3}$,

As Variance $= \sum X^2P(X) - (\text{Mean})^2$

$\therefore$ Variance $= 0^2\times\frac{625}{1296} + 1^2\times\frac{500}{1296} + 2^2\times\frac{150}{1296} + 3^2\times\frac{20}{1296} + 4^2\times\frac{1}{1296} - \left(\frac{2}{3}\right)^2$

$\Rightarrow$ Variance $= \frac{5}{9}$.

Q22. Let $f : N \to R$ be a function defined as $f(x) = 4x^2 + 12x + 15$. Show that $f : N \to S$, where S is the range of f, is invertible. Also find the inverse of f.

Sol. Here $f : N \to R,\ f(x) = 4x^2 + 12x + 15$

Let y be an arbitrary element of range S of function f. Then $y = 4x^2 + 12x + 15$, for some x in N, which implies that $y = (2x + 3)^2 + 6$.

This gives $x = \dfrac{\sqrt{y-6}-3}{2}$ as $y \geq 6$.

Let us define $g : S \to N$ by $g(y) = \dfrac{\sqrt{y-6}-3}{2}$.

Now, $gof(x) = g\left(f(x)\right) = g(4x^2 + 12x + 15) = g\left((2x+3)^2 + 6\right) = \dfrac{\sqrt{\left((2x+3)^2+6\right)-6}-3}{2} = x$.

And, $fog(y) = f\left(g(y)\right) = f\left(\dfrac{\sqrt{y-6}-3}{2}\right) = \left(2\left(\dfrac{\sqrt{y-6}-3}{2}\right)+3\right)^2 + 6 = y$.

Hence, $gof = I_N$ and $fog = I_S$. This implies that f is invertible with $f^{-1} = g$.

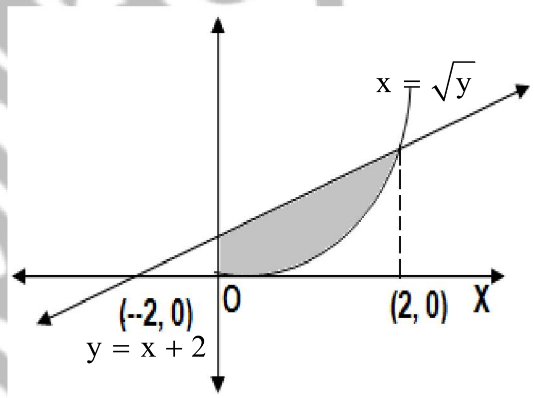
So, $f^{-1} = \dfrac{\sqrt{y-6}-3}{2}$ i.e., $f^{-1}(x) = \dfrac{\sqrt{x-6}-3}{2}$.

Q23. Using integration, find the area of the region bounded by the line $x - y + 2 = 0$, the curve $x = \sqrt{y}$ and y-axis.

Sol. We have $x - y + 2 = 0 \ldots$(i) and $x = \sqrt{y} \ldots$(ii)

Solving (i) & (ii), $x^2 = x + 2 \ \Rightarrow x^2 - x - 2 = 0$

$\Rightarrow (x-2)(x+1) = 0 \quad \therefore x = 2,\ \because x = -1$



Required area $= \int_0^2 \left[y_{(i)} - y_{(ii)}\right] dx$

$\Rightarrow \quad = \int_0^2 \left[x + 2 - x^2\right] dx$

$\Rightarrow \quad = \left[\dfrac{1}{2}x^2 + 2x - \dfrac{1}{3}x^3\right]_0^2$

$\Rightarrow \quad = \left[2 + 4 - \dfrac{8}{3} - 0\right] = \dfrac{10}{3}$ sq.units.

Q24. Find the distance of the point P(1, –2, 3) from the plane $x - y + z = 5$ measured parallel to the line whose direction cosines are proportional to 2, 3, –6.

Sol. Equation of a line through P(1,–2,3) & parallel to a line whose d.r.'s are proportional to 2, 3, –6 is : $\dfrac{x-1}{2} = \dfrac{y+2}{3} = \dfrac{z-3}{-6} = \lambda \ldots$(i).

Any random point on line (i) is $Q(2\lambda + 1, 3\lambda - 2, 3 - 6\lambda)$.

If Q lies on the given equation of plane $x - y + z = 5$ then,

$(2\lambda + 1) - (3\lambda - 2) + (3 - 6\lambda) = 5 \qquad \Rightarrow \lambda = \dfrac{1}{7}$.

So, coordinates of the point Q are $Q\left(\dfrac{2}{7}+1, \dfrac{3}{7}-2, 3-\dfrac{6}{7}\right)$ i.e., $Q\left(\dfrac{9}{7}, -\dfrac{11}{7}, \dfrac{15}{7}\right)$.

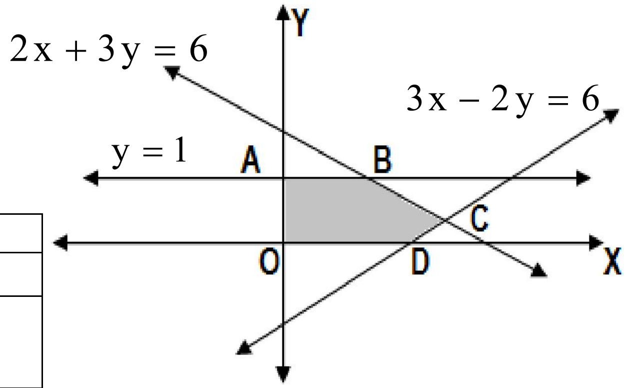
Required distance, $PQ = \sqrt{\left(\dfrac{9}{7}-1\right)^2 + \left(-\dfrac{11}{7}+2\right)^2 + \left(\dfrac{15}{7}-3\right)^2} = 1$ unit.

Q25. Maximise $z = 8x + 9y$, subject to the constraints given below :

$2x + 3y \leq 6$,

$3x-2y\le 6$,
$y\le 1$,
$x, y\ge 0$.

Sol. To Maximise $z=8x+9y$,
Subject to the constraints given below :
$2x+3y\le 6$,
$3x-2y\le 6$,
$y\le 1$; $x, y\ge 0$.



Corner points	Value of z
A(0,1)	9
$B\left(\frac{3}{2},1\right)$	21
$C\left(\frac{30}{13},\frac{6}{13}\right)$	$\frac{294}{13}=22\frac{8}{13}\leftarrow$ Max. value
D(2,0)	16
O(0, 0)	0

So maximum value of z is attained at $\left(\frac{30}{13},\frac{6}{13}\right)$ and maximum value is $22\frac{8}{13}$.

Q26. Find the minimum value of $(ax+by)$, where $xy=c^2$.

OR Find the coordinates of a point of the parabola $y=x^2+7x+2$ which is closest to the straight line $y=3x-3$.

Sol. Given $xy=c^2$...(i)

Let $S=(ax+by)$ $\quad\Rightarrow S=ax+\frac{bc^2}{x}$

$\Rightarrow\frac{dS}{dx}=a-\frac{bc^2}{x^2}$ and, $\frac{d^2S}{dx^2}=\frac{2bc^2}{x^3}$

For local points of maxima and/or minima, $\frac{dS}{dx}=a-\frac{bc^2}{x^2}=0$

$\Rightarrow x=c\sqrt{\frac{b}{a}}$

$\because \left.\frac{d^2S}{dx^2}\right]_{\text{at } x=c\sqrt{\frac{b}{a}}}=\frac{2bc^2}{c^3\left(\frac{b}{a}\right)^{3/2}}>0$

$\therefore$ S is minimum at $x=c\sqrt{\frac{b}{a}}$

Also, minimum value of $S=ax+by=2ax$

$$\left[\begin{array}{l}\because x=c\sqrt{\frac{b}{a}}\Rightarrow c^2=\frac{ax^2}{b}\\ \text{Replacing value of } c^2 \text{ in (i), we get } ax=by\end{array}\right]$$

That is, $S=2ac\sqrt{\frac{b}{a}}$

$\therefore S=2c\sqrt{ab}$.

OR Given line is $y=3x-3$ i.e., $3x-y-3=0$ ...(i)

Also let the required point on the parabola $y = x^2 + 7x + 2$ be P(h, k)

$\therefore k = h^2 + 7h + 2 ...(i)$

Distance of P from line 3x – y – 3 = 0 is, $s = \frac{|3h - k - 3|}{\sqrt{3^2 + (-1)^2}}$

$\Rightarrow s = \frac{|3h - h^2 - 7h - 2 - 3|}{\sqrt{10}}$, by (i)

$\Rightarrow s = \frac{|-h^2 - 4h - 5|}{\sqrt{10}} = \frac{|h^2 + 4h + 5|}{\sqrt{10}}$

$\Rightarrow s = \frac{(h+2)^2 + 1}{\sqrt{10}}$

$\Rightarrow \frac{ds}{dh} = \frac{2(h+2) + 0}{\sqrt{10}} = \frac{2(h+2)}{\sqrt{10}}$

and, $\frac{d^2s}{dh^2} = \frac{2}{\sqrt{10}} > 0$

So, s is least for all value of h.

For local points of maxima and/or minima, $\frac{ds}{dh} = \frac{2(h+2)}{\sqrt{10}} = 0$

$\Rightarrow h = -2$

By (i), $k = (-2)^2 + 7(-2) + 2 = -8$

Hence the coordinates of the required points on the given parabola are P(–2, –8).